FUNCTIONAL INEQUALITIES INVOLVING MODIFIED STRUVE FUNCTIONS

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Abstract. In this paper our aim is to prove some monotonicity and convexity results for the modified Struve function of the second kind by using its integral representation. Moreover, as consequences of these results, we present some functional inequalities (like Turán type inequalities) as well as lower and upper bounds for modified Struve function of the second kind and its logarithmic derivative.

1. Introduction

In the last decades many functional inequalities and monotonicity properties for special functions (like Bessel, modified Bessel, Gaussian hypergeometric, Kummer hypergeometric) and their several combinations have been deduced by many researchers, motivated by several problems that arise in wave mechanics, fluid mechanics, electrical engineering, quantum billiards, biophysics, mathematical physics, finite elasticity, probability and statistics, special relativity and radar signal processing. Although the inequalities involving the quotients of modified Bessel functions of the first and second kind are interesting in their own right, recently the lower and upper bounds for such quotients received increasing attention, since they play an important role in various problems of mathematical physics and electrical engineering. For more details, see for example [3] and the references therein. The modified Struve functions of the first and second kind are related to modified Bessel functions of the first kind, thus their properties can be useful in problems of mathematical physics. In [6] Joshi and Nalwaya presented some two-sided inequalities for modified Struve functions of the first kind and for their ratios. They deduced also some Turán and Wronski type inequalities for modified Struve functions of the first kind by using some generalized hypergeometric function representation of the Cauchy product of two modified Struve functions of the first kind. Motivated by the above results, by using a classical result on the monotonicity of quotients of MacLaurin series, recently in [5] we proved some monotonicity and convexity results for the modified Struve functions of the first kind. Moreover, as consequences of these results, we presented some functional inequalities as well as lower and upper bounds for modified Struve functions of the first kind. In this paper our aim is to continue the study from [5] for the modified Struve functions of the second kind. The key tools in the proofs of the main results are the techniques developed in the extensive study of modified Bessel functions of the first and second kind and their ratios. The difficulty in the study of the modified Struve function consists in the fact that the modified Struve differential equation is not homogeneous, however, as we can see below, the integral representation of modified Struve function of the second kind is very useful in order to study its monotonicity and convexity properties.

2. Modified Struve function: Monotonicity patterns and functional inequalities

The modified Struve functions of the first and second kind, $\mathbf{L}_\nu$ and $\mathbf{M}_\nu$ are particular solutions of the modified Struve equation [8, p. 288]

$$x^2y''(x)+xy'(x)-(x^2+\nu^2)y(x)=\frac{x^{\nu+1}}{\sqrt{\pi}2^{\nu-1}\Gamma\left(\nu+\frac{1}{2}\right)}. \tag{2.1}$$

The modified Struve function of the second kind has its power series representation as

$$\mathbf{M}_\nu(x)=\mathbf{L}_\nu(x)-I_\nu(x)=\sum_{n\geq0}\frac{\left(\frac{x}{2}\right)^{2n+\nu+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)}-\sum_{n\geq0}\frac{\left(\frac{x}{2}\right)^{2n+\nu}}{\Gamma\left(n+1\right)\Gamma\left(n+\nu+1\right)},$$

where I_ν stands for the modified Bessel function of the first kind.

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Now, let us consider the function $\mathcal{M}_\nu : (0,\infty) \to \mathbb{R}$, defined by

$$\mathcal{M}_\nu(x) = -2^\nu \Gamma\left(\nu + \frac{1}{2}\right) x^{-\nu} \mathbf{M}_\nu(x), \tag{2.2}$$

which for $\nu > -\frac{1}{2}$ has the integral representation [8, p. 292]

$$\mathcal{M}_\nu(x) = \frac{2}{\sqrt{\pi}} \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} e^{-xt} \mathrm{d}t \tag{2.3}$$

Our main result is the following theorem.

Theorem 1. *The following assertions are true:*

- **a.** *The function $x \mapsto \mathcal{M}_\nu(x)$ is completely monotonic and log-convex on $(0,\infty)$ for all $\nu > -\frac{1}{2}$.*
- **b.** *The function $\nu \mapsto \mathcal{M}_\nu(x)$ is completely monotonic and log-convex on $\left(-\frac{1}{2},\infty\right)$ for all $x > 0$.*
- **c.** *The function $x \mapsto -\mathbf{M}_\nu(x)$ is completely monotonic and log-convex on $(0,\infty)$ for all $\nu \in \left[-\frac{1}{2}, 0\right]$.*

Consequently, for all $x > 0$ the following inequalities are valid:

$$\mathcal{M}_\nu(x) \le \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)}, \quad \nu > -\frac{1}{2}, \tag{2.4}$$

$$0 < \left[\mathbf{M}_\nu(x)\right]^2 - \mathbf{M}_{\nu-1}(x)\mathbf{M}_{\nu+1}(x) < \frac{\left[\mathbf{M}_\nu(x)\right]^2}{\nu+\frac{1}{2}}, \quad \nu > \frac{1}{2}, \tag{2.5}$$

$$\frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)} < \nu, \quad \nu > -\frac{1}{2}, \tag{2.6}$$

$$-\sqrt{x^2+\nu^2} < \frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)} < \sqrt{x^2+\nu^2}, \quad \nu > \frac{1}{2}, \tag{2.7}$$

Proof. a. & **b.** Observe that for $n, m \in \{0,1,\dots\}$ and $\nu > -\frac{1}{2}$ we have

$$(-1)^n \left[\mathcal{M}_\nu(x)\right]^{(n)} = \frac{2}{\sqrt{\pi}} \int_0^1 t^n \left(1-t^2\right)^{\nu-\frac{1}{2}} e^{-xt} \mathrm{d}t,$$

$$(-1)^m \frac{\partial^m \mathcal{M}_\nu(x)}{\partial \nu^m} = \frac{2}{\sqrt{\pi}} \int_0^1 \left(\log\frac{1}{1-t^2}\right)^m \left(1-t^2\right)^{\nu-\frac{1}{2}} e^{-xt} \mathrm{d}t.$$

These can be proved easily by induction on n and m. Thus, the functions $x \mapsto \mathcal{M}_\nu(x)$ and $\nu \mapsto \mathcal{M}_\nu(x)$ are indeed completely monotonic and consequently are log-convex, since every completely monotonic function is log-convex, see [12, p. 167]. Alternatively, the log-convexity of these functions can be proved also by using (2.3) and the Hölder-Rogers inequality for integrals. Moreover, we arrive at the same conclusion by noticing that the integrand of (2.3) is log-convex in x and ν, and the integral preserves the log-convexity.

For inequality (2.4) just observe that $\mathcal{M}_\nu$ is decreasing on $(0,\infty)$ for all $\nu > -\frac{1}{2}$, and thus

$$\mathcal{M}_\nu(x) < \frac{2}{\sqrt{\pi}} \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \mathrm{d}t = \frac{1}{\sqrt{\pi}} \int_0^1 s^{-\frac{1}{2}}(1-s)^{\nu-\frac{1}{2}} \mathrm{d}s = \frac{1}{\sqrt{\pi}} B\left(\frac{1}{2}, \nu+\frac{1}{2}\right),$$

where $B(\cdot,\cdot)$, expressed as $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, stands for the Euler's beta function.

c. From (2.2) clearly we have

$$-\mathbf{M}_\nu(x) = \frac{x^\nu \mathcal{M}_\nu(x)}{2^\nu \Gamma\left(\nu+\frac{1}{2}\right)}.$$

On the other hand, observe that $x \mapsto x^\nu$ is completely monotonic on $(0,\infty)$ for all $\nu \le 0$. Thus, by using part **a** of this theorem we have that the function $x \mapsto -\mathbf{M}_\nu(x)$, as a product of two completely monotonic functions, is completely monotonic and log-convex on $(0,\infty)$ for all $\nu \in \left(-\frac{1}{2}, 0\right]$. Now, since

$$\mathbf{M}_{-\frac{1}{2}}(x) = \mathbf{L}_{-\frac{1}{2}}(x) - I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sinh x - \sqrt{\frac{2}{\pi x}} \cosh x = -\sqrt{\frac{2}{\pi x}} e^{-x},$$

the function $x \mapsto -\mathbf{M}_{-\frac{1}{2}}(x)$ is clearly completely monotonic and log-convex as a product of the completely monotonic and log-convex functions $x \mapsto \sqrt{\frac{2}{\pi}} x^{-\frac{1}{2}}$ and $x \mapsto e^{-x}$.

Now, let us focus on the Turán type inequality (2.5). Since $\nu \mapsto \mathcal{M}_\nu(x)$ is log-convex on $\left(-\frac{1}{2},\infty\right)$ for $x>0$, it follows that for all $\nu_1,\nu_2>-\frac{1}{2}$, $\alpha\in[0,1]$ and $x>0$ we have

$$\mathcal{M}_{\alpha\nu_1+(1-\alpha)\nu_2}(x) \le \left[\mathcal{M}_{\nu_1}(x)\right]^{\alpha}\left[\mathcal{M}_{\nu_2}(x)\right]^{1-\alpha}.$$

Choosing $\nu_1=\nu-1$, $\nu_2=\nu+1$ and $\alpha=\frac{1}{2}$, the above inequality reduces to the Turán type inequality

$$\left[\mathcal{M}_\nu(x)\right]^2-\mathcal{M}_{\nu-1}(x)\mathcal{M}_{\nu+1}(x)\le 0,$$

which is equivalent to the right-hand side of (2.5). For the left-hand side (2.5) observe that the Turánian

$${}_{\mathbf{M}}\Delta_\nu(x)=\left[\mathbf{M}_\nu(x)\right]^2-\mathbf{M}_{\nu-1}(x)\mathbf{M}_{\nu+1}(x)$$

can be rewritten as

$$ {}_{\mathbf{M}}\Delta_\nu(x)={}_{I}\Delta_\nu(x)+{}_{\mathbf{L}}\Delta_\nu(x)+{}_{I,\mathbf{L}}\Delta_\nu(x), \tag{2.8}$$

where

$$\begin{aligned}{}_{I}\Delta_\nu(x)&=\left[I_\nu(x)\right]^2-I_{\nu-1}(x)I_{\nu+1}(x),\\ {}_{\mathbf{L}}\Delta_\nu(x)&=\left[\mathbf{L}_\nu(x)\right]^2-\mathbf{L}_{\nu-1}(x)\mathbf{L}_{\nu+1}(x)\end{aligned}$$

and

$${}_{I,\mathbf{L}}\Delta_\nu(x)=I_{\nu+1}(x)\mathbf{L}_{\nu-1}(x)+I_{\nu-1}(x)\mathbf{L}_{\nu+1}(x)-2I_\nu(x)\mathbf{L}_\nu(x).$$

It is well-known that ${}_{I}\Delta_\nu(x)>0$ for all $\nu>-1$ and $x>0$, and ${}_{\mathbf{L}}\Delta_\nu(x)>0$ for all $\nu>-\frac{3}{2}$ and $x>0$. For these Turán type inequalities see for example [1, 2, 4, 5, 6, 11]. On the other hand, by using the integral representations

$$\begin{aligned}I_\nu(x)&=\frac{2\left(\frac{1}{2}x\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\cosh(xt)\mathrm{d}t,\\ \mathbf{L}_\nu(x)&=\frac{2\left(\frac{1}{2}x\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\sinh(xt)\mathrm{d}t,\end{aligned}$$

where $\nu>-\frac{1}{2}$, and the relation $\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu-\frac{1}{2}\right)=\Gamma^2\left(\nu+\frac{1}{2}\right)$ we obtain for all $\nu>\frac{1}{2}$ and $x>0$ that

$$\begin{aligned}{}_{I,\mathbf{L}}\Delta_\nu(x)=&\frac{4\left(\frac{1}{2}x\right)^{2\nu}}{\pi\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu-\frac{1}{2}\right)}\left[\int_0^1\int_0^1(1-t^2)^{\nu+\frac{1}{2}}(1-s^2)^{\nu-\frac{3}{2}}\cosh(xt)\sinh(xs)\mathrm{d}t\mathrm{d}s\right.\\ &\left.+\int_0^1\int_0^1(1-t^2)^{\nu-\frac{3}{2}}(1-s^2)^{\nu+\frac{1}{2}}\cosh(xt)\sinh(xs)\mathrm{d}t\mathrm{d}s\right]\\ &-\frac{8\left(\frac{1}{2}x\right)^{2\nu}}{\pi\Gamma^2\left(\nu+\frac{1}{2}\right)}\int_0^1\int_0^1(1-t^2)^{\nu-\frac{1}{2}}(1-s^2)^{\nu-\frac{1}{2}}\cosh(xt)\sinh(xs)\mathrm{d}t\mathrm{d}s\\ =&\frac{4\left(\frac{1}{2}x\right)^{2\nu}}{\pi\Gamma^2\left(\nu+\frac{1}{2}\right)}\int_0^1\int_0^1(1-t^2)^{\nu-\frac{3}{2}}(1-s^2)^{\nu-\frac{3}{2}}\left[(1-t^2)^2+(1-s^2)^2\right.\\ &\left.-2(1-t^2)(1-s^2)\right]\cosh(xt)\sinh(xs)\mathrm{d}t\mathrm{d}s\\ =&\frac{4\left(\frac{1}{2}x\right)^{2\nu}}{\pi\Gamma^2\left(\nu+\frac{1}{2}\right)}\int_0^1\int_0^1(1-t^2)^{\nu-\frac{3}{2}}(1-s^2)^{\nu-\frac{3}{2}}(t^2-s^2)^2\cosh(xt)\sinh(xs)\mathrm{d}t\mathrm{d}s.\end{aligned}$$

This shows that ${}_{I,\mathbf{L}}\Delta_\nu(x)>0$ for all $\nu>\frac{1}{2}$ and $x>0$, and consequently in view of (2.8) we have that ${}_{\mathbf{M}}\Delta_\nu(x)>0$ for all $\nu>\frac{1}{2}$ and $x>0$, as we required.

Next we prove the inequalities (2.6) and (2.7). Since for $\nu>-\frac{1}{2}$ the function $\mathcal{M}_\nu$ is completely monotonic on $(0,\infty)$, it follows that it is decreasing on $(0,\infty)$ for $\nu>-\frac{1}{2}$. Consequently, the function $x\mapsto\log\left(-x^{-\nu}\mathbf{M}_\nu(x)\right)$ is also decreasing on $(0,\infty)$ for $\nu>-\frac{1}{2}$, which in turn implies the inequality (2.6). Now, we show that the inequalities (2.5) and (2.6) imply the inequality (2.7). For this first observe that if we use the recurrence relations (see [8, p. 251] and [8, p. 292]) for the functions $\mathbf{L}_\nu$ and I_ν it can be shown that the function $\mathbf{M}_\nu$ satisfies the same recurrence relations as $\mathbf{L}_\nu$, that is,

$$\mathbf{M}_{\nu-1}(x)-\mathbf{M}_{\nu+1}(x)=\frac{2\nu}{x}\mathbf{M}_\nu(x)+\frac{\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}, \tag{2.9}$$

$$\mathbf{M}_{\nu-1}(x)+\mathbf{M}_{\nu+1}(x)=2\mathbf{M}_\nu'(x)-\frac{\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)},$$

$$x\mathbf{M}_\nu'(x)+\nu\mathbf{M}_\nu(x)=x\mathbf{M}_{\nu-1}(x). \tag{2.10}$$

Now, combining the recurrence relations (2.9) and (2.10) we obtain

$$\mathbf{M}_{\nu+1}(x)=\mathbf{M}_\nu'(x)-\frac{\nu}{x}\mathbf{M}_\nu(x)-\frac{\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{3}{2}\right)}, \tag{2.11}$$

and consequently, by using (2.10) and (2.11) we get

$$_{\mathbf{M}}\Delta_\nu(x)=\left(1+\frac{\nu^2}{x^2}\right)[\mathbf{M}_\nu(x)]^2-[\mathbf{M}_\nu'(x)]^2+\frac{x^\nu\mathbf{M}_{\nu-1}(x)}{\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)}.$$

But, according to the left-hand side of (2.5) we have $_{\mathbf{M}}\Delta_\nu(x)>0$ for $x>0$ and $\nu>\frac{1}{2}$, and consequently

$$\left(1+\frac{\nu^2}{x^2}\right)[\mathbf{M}_\nu(x)]^2-[\mathbf{M}_\nu'(x)]^2>-\frac{x^\nu\mathbf{M}_{\nu-1}(x)}{\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)}>0,$$

that is, for $x>0$ and $\nu>\frac{1}{2}$ we have

$$\left(\frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)}-\sqrt{x^2+\nu^2}\right)\left(\frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)}+\sqrt{x^2+\nu^2}\right)<0.$$

The inequality (2.6) clearly implies the right-hand side of (2.7), and this together with the above inequality imply that the left-hand side of (2.7) is valid. □

Concluding remarks and further results.

1. From (2.4) and part **a** of Theorem 1 it is clear that the function $x\mapsto\frac{\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{1}{2}\right)}\mathcal{M}_\nu(x)$ maps $(0,\infty)$ into $(0,1)$ and it is completely monotonic on $(0,\infty)$ for all $\nu>-\frac{1}{2}$. On the other hand, according to Kimberling [7] it is known that if the function f, defined on $(0,\infty)$, is continuous and completely monotonic and maps $(0,\infty)$ into $(0,1)$, then $\log f$ is super-additive, that is for all $x,y>0$ we have

$$\log f(x+y)\geq\log f(x)+\log f(y)\quad\text{or}\quad f(x+y)\geq f(x)f(y).$$

By using this result for all $x,y>0$ and $\nu>-\frac{1}{2}$ we have the inequality

$$\mathcal{M}_\nu(x+y)\geq\frac{\Gamma(\nu+1)}{\Gamma\left(\nu+\frac{1}{2}\right)}\mathcal{M}_\nu(x)\mathcal{M}_\nu(y).$$

2. We note that it is possible to complement and improve the inequality (2.4). Namely, it can be shown that the inequality

$$\mathcal{M}_\nu(x)\geq\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)}\cdot\frac{1-e^{-x}}{x} \tag{2.12}$$

is valid for $\nu\geq\frac{1}{2}$ and $x>0$. Moreover, the above inequality is reversed when $|\nu|<\frac{1}{2}$ and $x>0$, and by means of the inequality $e^{-x}>1-x$, the reversed form of the inequality (2.12) is better than (2.4) for $|\nu|<\frac{1}{2}$ and $x>0$. Now, let us recall the Chebyshev integral inequality [9, p. 40], which is as follows: if $f,g:[a,b]\to\mathbb{R}$ are integrable functions, both increasing or both decreasing and $p:[a,b]\to\mathbb{R}$ is a positive integrable function, then

$$\int_a^b p(t)f(t)\mathrm{d}t\int_a^b p(t)g(t)\mathrm{d}t\leq\int_a^b p(t)\mathrm{d}t\int_a^b p(t)f(t)g(t)\mathrm{d}t. \tag{2.13}$$

Note that if one of the functions f or g is decreasing and the other is increasing, then (2.13) is reversed. Now, we shall use (2.13) and (2.3) to prove (2.12). For this consider the functions $p,f,g:[0,1]\to\mathbb{R}$, defined by

$$p(t)=1,\ f(t)=\frac{2}{\sqrt{\pi}}(1-t^2)^{\nu-\frac{1}{2}}\ \text{ and }\ g(t)=e^{-xt}.$$

Observe that g is decreasing and f is increasing (decreasing) if $-\frac{1}{2}<\nu\leq\frac{1}{2}$ $(\nu\geq\frac{1}{2})$. On the other hand, we have

$$\mathcal{M}_\nu(0)=\frac{2}{\sqrt{\pi}}\int_0^1\left(1-t^2\right)^{\nu-\frac{1}{2}}\mathrm{d}t=\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)}\quad\text{and}\quad\int_0^1 e^{-xt}\mathrm{d}t=\frac{1-e^{-x}}{x},$$

and applying the Chebyshev inequality (2.13) we get the inequality (2.12) for all $\nu\geq\frac{1}{2}$, and its reverse when $|\nu|<\frac{1}{2}$.

3. It is also important to note here that by using the Chebyshev integral inequality (2.13) it is possible to derive more inequalities for the modified Struve functions. For example, if we consider the functions $p, f, g : [0, 1] \to \mathbb{R}$, defined by

$$p(t) = e^{-xt},\ f(t) = \frac{2}{\sqrt{\pi}}(1-t^2)^{\nu-\frac{3}{2}} \text{ and } g(t) = \frac{2}{\sqrt{\pi}}(1-t^2)^{\nu+\frac{1}{2}},$$

and we take into account the relation

$$\mathbf{M}_{\frac{1}{2}}(x) = \mathbf{L}_{\frac{1}{2}}(x) - I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}(\cosh x - 1) - \sqrt{\frac{2}{\pi x}}\sinh x = \sqrt{\frac{2}{\pi x}}(e^{-x} - 1),$$

then we get the inequality

$$\mathcal{M}_{\nu-1}(x)\mathcal{M}_{\nu+1}(x) \leq \mathcal{M}_{\frac{1}{2}}(x)\mathcal{M}_{2\nu-\frac{1}{2}}(x)$$

or equivalently

$$\mathbf{M}_{\nu-1}(x)\mathbf{M}_{\nu+1}(x) \leq \frac{\Gamma(2\nu)}{\Gamma\left(\nu-\frac{1}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}\mathbf{M}_{\frac{1}{2}}(x)\mathbf{M}_{2\nu-\frac{1}{2}}(x).$$

These inequalities are valid for $\nu \geq \frac{3}{2}$ and $x > 0$, and are reversed when $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $x > 0$.

4. Next we show that it is possible to deduce similar inequalities to those given in (2.7) in the case when $\nu \in \left[-\frac{1}{2}, 0\right]$. Namely, according to part **c** of Theorem 1, the function $x \mapsto -\mathbf{M}_\nu(x)$ is completely monotonic and log-convex on $(0,\infty)$ for all $\nu \in \left[-\frac{1}{2}, 0\right]$, and consequently we have

$$\mathbf{M}_\nu''(x)\mathbf{M}_\nu(x) - \left[\mathbf{M}_\nu'(x)\right]^2 > 0$$

for all $x > 0$ and $\nu \in \left[-\frac{1}{2}, 0\right]$. On the other hand, recall that the modified Struve function $\mathbf{M}_\nu$ is a particular solution of the modified Struve equation (2.1) and consequently

$$\mathbf{M}_\nu''(x) = \left(1 + \frac{\nu^2}{x^2}\right)\mathbf{M}_\nu(x) - \frac{1}{x}\mathbf{M}_\nu'(x) + \frac{x^{\nu-1}}{\sqrt{\pi}2^{\nu-1}\Gamma\left(\nu+\frac{1}{2}\right)}.$$

Combining this with the above inequality we get that

$$\left(1 + \frac{\nu^2}{x^2}\right)\left[\mathbf{M}_\nu(x)\right]^2 - \frac{1}{x}\mathbf{M}_\nu(x)\mathbf{M}_\nu'(x) - \left[\mathbf{M}_\nu'(x)\right]^2 > 0,$$

that is,

$$\left[\frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)}\right]^2 + \frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)} - (x^2+\nu^2) < 0.$$

Here we used the fact that $\mathbf{M}_\nu(x) < 0$ for $x > 0$ and $\nu \geq -\frac{1}{2}$. Now, taking into account the above inequality we clearly have

$$\frac{-1-\sqrt{1+4(x^2+\nu^2)}}{2} < \frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)} < \frac{-1+\sqrt{1+4(x^2+\nu^2)}}{2} \tag{2.14}$$

for all $x > 0$ and $\nu \in \left[-\frac{1}{2}, 0\right]$. Observe that the left-hand side of (2.14) is weaker than the left-hand side of (2.7), however, the right-hand side of (2.14) is better than the right-hand side of (2.7). Moreover, it is important to mention here that $\mathbf{M}_\nu'(x) > 0$ for $x > 0$ and $\nu \in \left[-\frac{1}{2}, 0\right]$, and consequently the expression $\dfrac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)}$ is negative, which clearly implies the right-hand side of (2.14).

5. Now, we will use the Fox–Wright generalized hypergeometric function ${}_p\Psi_q(\cdot)$, with p numerator and q denominator parameters, which is defined as the series

$${}_p\Psi_q\left[\begin{array}{c}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{array}\middle| z\right] = \sum_{n\geq 0}\frac{\prod_{l=1}^p\Gamma(a_l+\alpha_l n)}{\prod_{j=1}^q\Gamma(b_j+\beta_j n)}\frac{z^n}{n!}, \tag{2.15}$$

where $z, a_l, b_j \in \mathbb{C}$, $\alpha_l, \beta_j \in \mathbb{R}$ for $l \in \{1,\dots,p\}$ and $j \in \{1,\dots,q\}$. The series (2.15) converges absolutely and uniformly for all bounded $|z|$, $z \in \mathbb{C}$ when

$$\varepsilon = 1 + \sum_{j=1}^q \beta_j - \sum_{l=1}^p \alpha_l > 0\,.$$

Thus, by using (2.3) we have

$$\frac{\sqrt{\pi}}{2}\,\mathcal{M}_\nu(x)=\int_0^\infty(1-t^2)^{\nu-\frac12}\,e^{-xt}\,\mathrm{d}t=\frac12\sum_{n\geq0}\frac{(-x)^n}{n!}\int_0^1 s^{\frac{n-1}{2}}(1-s)^{\nu-\frac12}\,\mathrm{d}s$$
$$=\frac{\Gamma(\nu+\frac12)}{2}\sum_{n\geq0}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n+\nu+1)}\frac{(-x)^n}{n!}=\frac{\Gamma(\nu+\frac12)}{2}\,{}_1\Psi_1\left[\begin{array}{c}(\frac12,\frac12)\\(\nu+1,1)\end{array}\middle|-x\right],$$

where, being $\varepsilon=\frac32$, the series converges for all $x>0$. Therefore, for all $x>0$ we have

$$\mathcal{M}_\nu(x)=\frac{\Gamma(\nu+\frac12)}{\sqrt{\pi}}\,{}_1\Psi_1\left[\begin{array}{c}(\frac12,\frac12)\\(\nu+1,1)\end{array}\middle|-x\right].$$

On the other hand, [10, Theorem 4] says that for certain conditions on the parameters a_l, $b_j\in\mathbb{R}$, α_l, $\beta_j\in\mathbb{R}$ for $l\in\{1,\dots,p\}$ and $j\in\{1,\dots,q\}$, the next exponential inequalities are valid for all $x\in\mathbb{R}$

$$\psi_0 e^{\psi_1\psi_0^{-1}|x|}\leq{}_p\Psi_q\left[\begin{array}{c}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{array}\middle|\,x\right]\leq\psi_0-(1-e^{|x|})\psi_1, \tag{2.16}$$

where

$$\psi_0=\frac{\prod_{j=1}^p\Gamma(a_j)}{\prod_{j=1}^q\Gamma(b_j)}\quad\text{and}\quad\psi_1=\frac{\prod_{j=1}^p\Gamma(a_j+\alpha_j)}{\prod_{j=1}^q\Gamma(b_j+\beta_j)}.$$

In our case we have

$$\psi_0 e^{\psi_1\psi_0^{-1}|x|}=\frac{\sqrt{\pi}}{\Gamma(\nu+1)}e^{\frac{|x|}{\sqrt{\pi}(\nu+1)}}\quad\text{and}\quad\psi_0-(1-e^{|x|})\psi_1=\frac{\sqrt{\pi}}{\Gamma(\nu+1)}-(1-e^{|x|})\frac{1}{\Gamma(\nu+2)},$$

and the conditions of [10, Theorem 4] can be simplified as $\nu\geq\frac{\sqrt{\pi}}{2}-2$ and $\nu\geq\nu_0=\frac{4-\pi}{\pi-2}\simeq 0.7519383938\dots$. Consequently, by applying (2.16), for all $\nu\geq\nu_0$ and $x>0$ we have

$$\frac{\Gamma\left(\nu+\frac12\right)}{\Gamma(\nu+1)}e^{-\frac{x}{\sqrt{\pi}(\nu+1)}}\leq\mathcal{M}_\nu(x)\leq\frac{\Gamma\left(\nu+\frac12\right)}{\Gamma(\nu+1)}-\frac{\Gamma\left(\nu+\frac12\right)}{\sqrt{\pi}\Gamma(\nu+2)}(1-e^{-x}).$$

We note that this inequality complements and improve (2.4) when $\nu\geq\nu_0$.

6. Observe also that

$$\frac1x\left[\mathbf{M}_\nu(x)\right]^2\left[\frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)}\right]'=\left(1+\frac{\nu^2}{x^2}\right)\left[\mathbf{M}_\nu(x)\right]^2-\left[\mathbf{M}_\nu'(x)\right]^2+\frac{\left(\nu+\frac12\right)x^{\nu-1}\mathbf{M}_\nu(x)}{\sqrt{\pi}2^{\nu-1}\Gamma\left(\nu+\frac32\right)}.$$

Thus, by using (2.6) and the fact that $\mathbf{M}_\nu(x)<0$ for $x>0$ and $\nu>-\frac12$, we have

$${}_{\mathbf{M}}\Delta_\nu(x)-\frac1x\left[\mathbf{M}_\nu(x)\right]^2\left[\frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)}\right]'=\frac{x^\nu\mathbf{M}_\nu(x)}{2^\nu\sqrt{\pi}\Gamma\left(\nu+\frac32\right)}\left[\frac{\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)}-\frac{\nu+1}{x}\right]>0.$$

Combining this with the right-hand side of the Turán type inequality (2.5) we obtain that for $x>0$ and $\nu>\frac12$ we have

$$\left[\frac{x\mathbf{M}_\nu'(x)}{\mathbf{M}_\nu(x)}\right]'<\frac{x}{\nu+\frac12}.$$

7. Finally we note that if we combine the inequalities [3, 5]

$$I_\nu(x)<\frac{x^\nu}{2^\nu\Gamma(\nu+1)}e^{\frac{x^2}{4(\nu+1)}}\quad\text{and}\quad\mathbf{L}_\nu(x)>\frac{x^\nu\sinh\frac{x}{2\nu+3}}{\sqrt{\pi}2^{\nu-1}\Gamma\left(\nu+\frac32\right)},$$

which hold for all $\nu>-1$ and $x>0$, then we obtain that

$$\mathbf{M}_\nu(x)>\frac{x^\nu\sinh\frac{x}{2\nu+3}}{\sqrt{\pi}2^{\nu-1}\Gamma\left(\nu+\frac32\right)}-\frac{x^\nu}{2^\nu\Gamma(\nu+1)}e^{\frac{x^2}{4(\nu+1)}}$$

is valid for all $\nu>-1$ and $x>0$, and consequently the inequality

$$\mathcal{M}_\nu(x)<\frac{\Gamma\left(\nu+\frac12\right)}{\Gamma(\nu+1)}e^{\frac{x^2}{4(\nu+1)}}-\frac{4}{\sqrt{\pi}(2\nu+1)}\sinh\frac{x}{2\nu+3}$$

is valid for all $x>0$ and $\nu>-\frac12$.

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